

Objectivity in the Photonic Environment Through State Information Broadcasting

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Recently, the emergence of classical objectivity as a property of a quantum state has been explicitly derived for a small object embedded in a photonic environment in terms of a *spectrum broadcast form*—a specific classically correlated state, redundantly encoding information about the preferred states of the object in the environment. However, the environment was in a pure state and the fundamental problem was how generic and robust is the conclusion. Here we prove that despite of the initial environmental noise the emergence of the broadcast structure still holds, leading to the perceived objectivity of the state of the object. We also show how this leads to a quantum Darwinism-type condition, reflecting classicality of proliferated information in terms of a limit behavior of the mutual information. Quite surprisingly, we find „singular points” of the decoherence, which can be used to faithfully broadcast a specific classical message through the noisy environment.

Keywords: decoherence, quantum darwinism, state broadcasting

Uninterrupted series of successes of quantum mechanics supports a belief that quantum formalism applies to all of physical reality. Thus, in particular, the objective classical world of everyday experience should emerge naturally from the formalism. This has been a long-standing problem, already present from the very dawn of quantum mechanics [1, 2]. Recently, a crucial step was made in a series of works (see e.g. [3–5]) introducing quantum Darwinism—a refined model of decoherence [6], based on a multiple environments paradigm: A quantum system of interest S interacts with multiple environments E_1, \dots, E_N instead of just one. The authors assumed [3] that each of these independent fractions effectively measures the system and argued that after the decoherence (with some timescale τ_D) it carries nearly complete classical information about the system, meaning that the information propagates in the environment with a huge redundancy. A further step was made in [7] by dropping any explicit assumptions on the dynamics and applying an operational definition of objectivity [4] directly to the post-decoherence quantum state. This, together with the Bohr’s criterion of non-disturbance [8], allowed to derive a universal state structure—*spectrum broadcast form* (cf. [9]), responsible for the appearance of classical objectivity in a model- and dynamics-independent way [7]:

There appears an objectively existing state of the system S if the time-asymptotic joint quantum state of S and the observed fraction of the environment fE is of a spectrum broadcast form:

$$\varrho_{S:fE}(\infty) = \sum_i p_i |\vec{x}_i\rangle \langle \vec{x}_i| \otimes \varrho_i^{E_1} \otimes \dots \otimes \varrho_i^{E_{fN}}, \quad \varrho_i^{E_k} \varrho_{i' \neq i}^{E_k} = 0, \quad (1)$$

with $\{|\vec{x}_i\rangle\}$ a pointer basis [10], p_i ’s initial pointer probabilities, and $\varrho_i^{E_1}, \dots, \varrho_i^{E_{fN}}$ some states of the environments E_1, \dots, E_{fN} with mutually orthogonal supports.

The states (1) "work" by faithfully encoding the same

classical information about the system (index i) in each portion of the environment—they describe redundant proliferation (broadcasting) of information, necessary for objectivity [4, 7]. A process of formation of a state (1) is what we call *state information broadcasting* [7]. It is a weaker form of the quantum state broadcasting [9, 11].

In this Letter we apply the above novel results to the celebrated model of a dielectric sphere illuminated by photons [12–16] to show how an objectively existing state of a system [4, 7] is actually formed in a course of the quantum evolution with a general (not only thermal) noisy environment. In contrast to the earlier studies [15, 16], we show it directly on the fundamental level of quantum states, proving the emergence of the broadcast structure (1), rather than using information-theoretical conditions, which so far are only known to necessary,

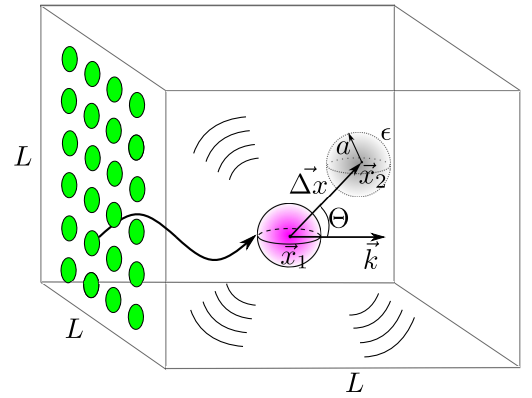


FIG. 1: The illuminated sphere model [12]. Green dots represent the photons, which constitute the environments E_1, \dots, E_N of the sphere. The sphere and the photons are enclosed in a large cubic box of edge L ; photon momentum eigenstates $|\vec{k}\rangle$ obey the periodic boundary conditions.

while their sufficiency is still not known [7]. We thus prove robustness and a generic character of the emergence of objectivity—a well known property of the everyday life. In other words, state information broadcasting process still works even if the environment is noisy, which in principle might cover or mismatch proliferation of emerging classical information about the system. Moreover, with a help of the classical Perron-Frobenius Theorem [17] we show a surprising effect of how the decoherence mechanism can be used to *faithfully broadcast* a specific message into the environment.

The model [12]. A dielectric sphere S of radius a and relative permittivity ϵ is bombarded by a constant flux of photons, constituting the sphere's environment; see Fig. 1. The sphere can be at two possible locations \vec{x}_1 and \vec{x}_2 and the photons are assumed not energetic enough to individually resolve the displacement $\Delta x \equiv |\vec{x}_2 - \vec{x}_1|$:

$$k\Delta x \ll 1, \quad (2)$$

where $\hbar k$ is some characteristic momentum, but they are able to do so collectively: If the sphere is initially in a superposition of the localized states $|\vec{x}_1\rangle$, $|\vec{x}_2\rangle$, the scattering photons will localize it via collisional decoherence [12]. Here we show that during this process a broadcast state (1) is formed for the radiation being initially a general mixture of plane waves, concentrated around (2):

$$\varrho_0^{ph} = \sum_{\vec{k}} p(\vec{k}) |\vec{k}\rangle \langle \vec{k}|, \quad \text{supp } p \in \{\vec{k} : |\vec{k}|\Delta x \ll 1\} \quad (3)$$

(in the previous studies only thermal states were considered [12, 15, 16]). Following [12, 13, 15, 16], we use box normalization to describe the photons; see Fig. 1. We remove it through the thermodynamic limit (signified by \cong) [15, 16]: $V \rightarrow \infty$, $N \rightarrow \infty$, $N/V = \text{const}$. The interaction time t enters through the number of scattered photons up to time t (a "macroscopic time"); see Fig. 1:

$$N_t \equiv L^2 \frac{N}{V} ct, \quad (4)$$

where c is the speed of light. We will work with a fixed t and pass to the decoherence limit $t/\tau_D \rightarrow \infty$ (denoted by \approx or ∞) at the very end. The sphere-photon interaction is of a controlled-unitary type (*symmetric environments*):

$$U_{S:E}(t) \equiv \sum_{i=1,2} |\vec{x}_i\rangle \langle \vec{x}_i| \otimes \underbrace{\mathbf{S}_i \otimes \cdots \otimes \mathbf{S}_i}_{N_t}, \quad (5)$$

where (assuming translational invariance) $\mathbf{S}_i \equiv \mathbf{S}_{\vec{x}_i} = e^{-i\vec{x}_i \cdot \vec{k}} \mathbf{S}_0 e^{i\vec{x}_i \cdot \vec{k}}$ is the scattering matrix (see e.g. [18]).

Macrofractions. We introduce a crucial *environment coarse-graining*: we divide the full photonic environment into a number of *macroscopic fractions*, each containing mN_t photons, $0 \leq m \leq 1$. By *macroscopic* we will understand "scaling with the total number of photons N_t ". By

definition, these are the environment fractions accessible to independent observers, searching for an objective state of the sphere [7]. In typical situations, detectors used to monitor the environment, e.g. eyes, have some minimum detection thresholds and the fractions mN_t are meant to reflect it. The concrete fraction size is irrelevant here—it is enough that it scales with N_t [19]. The detailed initial product state of the environment $(\varrho_0^{ph})^{\otimes N_t}$ can thus be trivially rewritten as:

$$\begin{aligned} \underbrace{\varrho_0^{ph} \otimes \cdots \otimes \varrho_0^{ph}}_{N_t} &= \underbrace{\varrho_0^{ph} \otimes \cdots \otimes \varrho_0^{ph}}_{mN_t} \otimes \cdots \otimes \underbrace{\varrho_0^{ph} \otimes \cdots \otimes \varrho_0^{ph}}_{mN_t} \\ &\equiv \underbrace{\varrho_0^{mac} \otimes \cdots \otimes \varrho_0^{mac}}_M, \end{aligned} \quad (6)$$

where $M \equiv 1/m$ is the number of macrofractions and $\varrho_0^{mac} \equiv (\varrho_0^{ph})^{\otimes mN_t}$ is the initial state of each of them.

Formation of the broadcast state. After all the N_t photons have scattered and $(1-f)M$, $0 \leq f \leq 1$, macrofractions went unobserved (the necessary loss of information), the post-scattering "out"-state $\varrho_{S:fE}(t) \equiv \text{Tr}_{(1-f)E} U_{S:E}(t) \varrho_{S:E}(0) U_{S:E}(t)^\dagger$, is given from (5,6) for a product initial state $\varrho_{S:E}(0) \equiv \varrho_0^S \otimes (\varrho_0^{ph})^{\otimes N_t}$ by:

$$\varrho_{S:fE}(t) = \sum_{i=1,2} \langle \vec{x}_i | \varrho_0^S | \vec{x}_i \rangle |\vec{x}_i\rangle \langle \vec{x}_i| \otimes [\varrho_i^{mac}(t)]^{\otimes fM} \quad (7)$$

$$\begin{aligned} &+ \sum_{i \neq j} \langle \vec{x}_i | \varrho_0^S | \vec{x}_j \rangle \left(\text{Tr} \mathbf{S}_i \varrho_0^{ph} \mathbf{S}_j^\dagger \right)^{(1-f)N_t} |\vec{x}_i\rangle \langle \vec{x}_j| \otimes \\ &\quad \otimes \left(\mathbf{S}_i \varrho_0^{ph} \mathbf{S}_j^\dagger \right)^{\otimes fN_t}, \end{aligned} \quad (8)$$

where $\varrho_i^{mac}(t) \equiv \left(\mathbf{S}_i \varrho_0^{ph} \mathbf{S}_i^\dagger \right)^{\otimes mN_t}$, $i = 1, 2$. We demonstrate that in the soft scattering sector (2), the above state approaches asymptotically the broadcast form (1) by showing that for $t \gg \tau_D$:

1. The post-scattering coherent part $\varrho_{S:fE}^{i \neq j}(t)$, defined by (8), vanishes in the trace norm (*decoherence*):

$$\|\varrho_{S:fE}^{i \neq j}(t)\|_{\text{tr}} \equiv \text{Tr} \sqrt{\left[\varrho_{S:fE}^{i \neq j}(t) \right]^\dagger \varrho_{S:fE}^{i \neq j}(t)} \approx 0. \quad (9)$$

2. The post-scattering macro-states $\varrho_i^{mac}(t)$ become *perfectly distinguishable*: $\varrho_1^{mac}(t) \varrho_2^{mac}(t) \approx 0$, or equivalently using the generalized overlap [21]:

$$\begin{aligned} B[\varrho_1^{mac}(t), \varrho_2^{mac}(t)] &\equiv \\ &\equiv \text{Tr} \sqrt{\sqrt{\varrho_1^{mac}(t)} \varrho_2^{mac}(t) \sqrt{\varrho_1^{mac}(t)}}} \approx 0, \end{aligned} \quad (10)$$

despite of the individual photon states (micro-states) becoming equal in the thermodynamic limit.

The first mechanism above is the usual decoherence of S by fE . Some form of quantum correlations may still

survive it, since the resulting state (7) is generally of a Classical-Quantum (CQ) form [22], but they are damped by the second mechanism and $\varrho_{S:fE}(\infty)$ becomes a spectrum broadcast state (1) for $p_i = \langle \vec{x}_i | \varrho_0^S \vec{x}_i \rangle$.

The decoherence mechanism alone (9) has been extensively studied in the model for thermal initial states ϱ_0^{ph} (see. e.g. [12–16]). We recall that the decay of the off-diagonal part $\varrho_{S:fE}^{i \neq j}(t)$, defined by (8), is governed by the decoherence factor $|\text{Tr} \mathbf{S}_1 \varrho_0^{ph} \mathbf{S}_2^\dagger|$, since $\|\varrho_{S:fE}^{i \neq j}(t)\|_{\text{tr}} = 2|\langle \vec{x}_1 | \varrho_0^S \vec{x}_2 \rangle| |\text{Tr} \mathbf{S}_1 \varrho_0^{ph} \mathbf{S}_2^\dagger|^{(1-f)N_t}$. For pure ϱ_0^{ph} , it reads in the regime (2) [12–16]:

$$\langle \vec{k} | \mathbf{S}_2^\dagger \mathbf{S}_1 \vec{k} \rangle = 1 + i \frac{8\pi \Delta x k^5 \tilde{a}^6}{3L^2} \cos \Theta - \frac{2\pi \Delta x^2 k^6 \tilde{a}^6}{15L^2} (3 + 11 \cos^2 \Theta) + O\left[\frac{(k\Delta x)^3}{L^2}\right], \quad (11)$$

where Θ is the angle between \vec{k} and $\vec{\Delta x} \equiv \vec{x}_2 - \vec{x}_1$, $\tilde{a} \equiv a[(\epsilon - 1)/(\epsilon + 2)]^{1/3}$, while for a general distribution (3) it is given in the leading order in $1/L$ by [12, 15, 16, 20]:

$$\begin{aligned} |\text{Tr} \mathbf{S}_1 \varrho_0^{ph} \mathbf{S}_2^\dagger|^{(1-f)N_t} &\cong \\ \left[1 - \frac{2\pi \Delta x^2 \tilde{a}^6}{15L^2} \sum_{\vec{k}} p(\vec{k}) k^6 (3 + 11 \cos^2 \Theta_{\vec{k}}) \right]^{(1-f)N_t} &\quad (12) \\ \xrightarrow{\text{therm.}} \exp\left[-\frac{(1-f)}{\overline{\tau_D}} t\right], &\quad (13) \end{aligned}$$

where $\overline{\tau_D}^{-1} \equiv \frac{2\pi}{15} \frac{N}{V} \Delta x^2 c \tilde{a}^6 \int d\vec{k} p(\vec{k}) k^6 (3 + 11 \cos^2 \Theta_{\vec{k}})$ is the decoherence time.

Completing the step (10) is more involved. For the micro-states $\varrho_i^{mic} \equiv \mathbf{S}_i \varrho_0^{ph} \mathbf{S}_i^\dagger$ we obtain under (2) [20]:

$$B(\varrho_1^{mic}, \varrho_2^{mic}) = 1 - \frac{\bar{\eta} - \bar{\eta}'}{L^2} \xrightarrow{\text{therm.}} 1, \quad (14)$$

where:

$$\begin{aligned} \bar{\eta} &\equiv \frac{L^2}{2} \left(1 - \sum_{\vec{k}} p(\vec{k}) \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k} \rangle \right|^2 \right) \cong \left(\frac{N}{\overline{\tau_D} V} c \right)^{-1} \quad (15) \\ \bar{\eta}' &\equiv \frac{L^2}{2} \sum_{\vec{k}} \sum_{\vec{k}' \neq \vec{k}} p(\vec{k}) \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle \right|^2, \quad (16) \end{aligned}$$

so that $\varrho_{1,2}^{mic}$ become equal and encode no information about S . Same holds if the observed portion μ of the environment E is *microscopic*, i.e. not scaling with N_t :

$$\begin{aligned} \varrho_{S:\mu E}(0) &= \varrho_0^S \otimes (\varrho_0^{mac})^{\otimes \mu} \xrightarrow[\text{therm.}]{t \gg \tau_D} \varrho_{S:\mu E}(\infty) = \\ &\left(\sum_{i=1,2} p_i |\vec{x}_i\rangle \langle \vec{x}_i| \right) \otimes [\varrho^{mic}]^{\otimes \mu}. \quad (17) \end{aligned}$$

This is a "product phase" [7], in which the mutual information $I[\varrho_{S:\mu E}(\infty)] = 0$.

Passing to macro-states the situation changes as now:

$$\begin{aligned} B[\varrho_1^{mac}(t), \varrho_2^{mac}(t)] &= \left(\text{Tr} \sqrt{\sqrt{\varrho_1^{mic}} \varrho_2^{mic} \sqrt{\varrho_1^{mic}}} \right)^{mN_t} \\ &\cong \left(1 - \frac{\alpha \bar{\eta}}{L^2} \right)^{mN_t} \xrightarrow{\text{therm.}} \exp\left[-\frac{\alpha m}{\overline{\tau_D}} t\right], \quad (18) \end{aligned}$$

where $\alpha \equiv (\bar{\eta} - \bar{\eta}')/\bar{\eta}$ [16] and (15) was used. Thus, whenever $\alpha \neq 0$ ($\alpha = 0$ e.g. for an isotropic illumination [20]), $B[\varrho_1^{mac}(t), \varrho_2^{mac}(t)] \approx 0$ for $t \gg \overline{\tau_D}/\alpha$, despite (14), i.e. the macro-states become perfectly distinguishable via orthogonal projectors on their supports. The latter are contained in $\text{span}\{|\vec{k}\rangle : \vec{k} \in \text{supp } p\}^{\otimes mN_t}$ (cf. (3)), rotated by $\mathbf{S}_1^{\otimes mN_t}$ and $\mathbf{S}_2^{\otimes mN_t}$ respectively. Eqs. (13,18) together imply an asymptotic formation of the spectrum broadcast state (1):

$$\begin{aligned} \varrho_{S:fE}(0) &= \varrho_0^S \otimes [\varrho_0^{mac}]^{\otimes fM} \xrightarrow[\text{therm.}]{t \gg \tau_D} \varrho_{S:fE}(\infty) = \\ &\sum_{i=1,2} \langle \vec{x}_i | \varrho_0^S \vec{x}_i \rangle |\vec{x}_i\rangle \langle \vec{x}_i| \otimes [\varrho_i^{mac}(\infty)]^{\otimes fM} \quad (19) \end{aligned}$$

with $\varrho_1^{mac}(\infty) \varrho_2^{mac}(\infty) = 0$ [23]. The scattering (19) is thus a combination of the localization measurement in the pointer basis $|\vec{x}_i\rangle$ and spectrum broadcasting of the result, described by a CC-type channel [9]:

$$\Lambda_{\infty}^{S \rightarrow fE}(\varrho_0^S) = \sum_i \langle \vec{x}_i | \varrho_0^S \vec{x}_i \rangle [\varrho_i^{mac}(\infty)]^{\otimes fM}. \quad (20)$$

As a consequence of (19), it follows that [20]:

$$I[\varrho_{S:fE}(t)] \xrightarrow[\text{therm.}]{t \gg \tau_D} H_S \text{ for any } 0 < f < 1, \quad (21)$$

i.e. the mutual information becomes asymptotically independent of the fraction size f (as long as it is macroscopic). This is the entropic objectivity condition of quantum Darwinism, leading to the characteristic classical plateau [4]. We stress that here (21) is *derived* as a consequence of the state information broadcasting (19) and we call this regime a "broadcasting phase" [7]. When the whole E is observed, modulo a micro-fraction, there appears from (8) a "full information phase", when quantum correlations are retained and $I[\varrho_{S:fE}(t)] \approx I_{max}$.

Comparing (13) and (18) we observe that, unlike in the pure case [7], the timescales of decoherence (9) and distinguishability (10) are a priori different (cf. [16]): $\overline{\tau_D}$ and $\overline{\tau_D}/\alpha$ respectively. Since $0 \leq \alpha \leq 1$, the broadcast state is fully formed for $t \gg \overline{\tau_D}/\alpha$. Environment noise thus slows down the formation of the broadcast state [24].

"Singular points" of decoherence. Let the initial state of the sphere have a diagonal representation $\varrho_0^S = \sum_i \lambda_{0i} |\phi_i\rangle \langle \phi_i|$. Then, in (19) there appears a stochastic matrix $P_{ij}(\phi) \equiv |\langle \phi_i | \vec{x}_j \rangle|^2$, which by the Perron-Frobenius Theorem [17] possesses at least one stable probability distribution $\lambda_{*i}(\phi)$: $\sum_j P_{ij}(\phi) \lambda_{*j}(\phi) =$

$\lambda_{*i}(\phi)$. It exists for *any* initial eigenbasis $|\phi_i\rangle$. Let us choose it as the initial spectrum: $\lambda_{0i} \equiv \lambda_{*i}(\phi)$. Then, the scattering (19) not only leaves this distribution unchanged, but broadcasts it into the environment:

$$\begin{aligned} & \left[\sum_i \lambda_{*i}(\phi) |\phi_i\rangle\langle\phi_i| \right] \otimes (\varrho_0^{mac})^{\otimes fM} \xrightarrow[\text{therm.}]{t \gg \tau_D} \varrho_{S:FE}(\infty) = \\ & = \sum_i \left(\sum_j P_{ij}(\phi) \lambda_{*j}(\phi) \right) |\vec{x}_i\rangle\langle\vec{x}_i| \otimes (\varrho_i^{mac})^{\otimes fM} \\ & = \sum_i \lambda_{*i}(\phi) |\vec{x}_i\rangle\langle\vec{x}_i| \otimes (\varrho_i^{mac})^{\otimes fM}. \end{aligned} \quad (22)$$

The initial spectrum does not “decohere”. This surprising Perron-Frobenius broadcasting [9], can thus be used to faithfully (in the asymptotic limit above) broadcast the classical message $\{\lambda_{*i}(\phi)\}$ through the environment macro-fractions, however noisy they are.

Final remarks. There is one straightforward generalization to many parties. Consider several spheres, each with its own photonic environment, separated by distances D , $kD \gg 1$ (cf. (2)). The interaction is then a product of (5), e.g. for two spheres $U_{S_1 S_2: E_1 E_2}(t) \equiv \sum_{i,j=1,2} |\vec{x}_i\rangle\langle\vec{x}_i| \otimes |\vec{y}_j\rangle\langle\vec{y}_j| \otimes \mathbf{S}_i^{\otimes N_t} \otimes \mathbf{S}_j^{\otimes N_t}$, where \vec{x}_i, \vec{y}_j are the spheres’ positions and $\mathbf{S}_i, \mathbf{S}_j$ are the corresponding scattering matrices. The asymptotic state (19) provides objectivisation of classical correlations [9], e.g. $p_{ij} \equiv \langle \vec{x}_i, \vec{y}_j | \varrho_0^S \vec{x}_i, \vec{y}_j \rangle$, measurable by observers who have an access to photons originating from all the spheres.

In the studied model, as in the majority of decoherence models, the system-environment interaction is of a form:

$$H_{int} = g A_S \sum_{k=1}^N X_{E_k}, \quad (23)$$

where g is a coupling constant and $A_S, X_{E_1}, \dots, X_{E_N}$ are some observables on the system and the environments respectively. The eigenbasis of $A = \sum_i a_i |i\rangle\langle i|$ becomes the pointer basis—it is arguably put by hand by the choice of the form (23). It is then an interesting question if there are more general interactions, without an a priori privileged basis, which nevertheless lead to an asymptotic formation of spectrum broadcast states (1).

Finally, it would be extremely interesting to test our findings experimentally. In fact, our central object, the broadcast state (1), is in principle directly observable through e.g. quantum state tomography—a well developed, successful, and widely used technique [25].

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- [23] By (13,18), the asymptotic formation of the spectrum broadcast state (19) relies on the full product form of the initial state $\varrho_{S:E}(0)$ and the interaction (5) in each block i . However, one can allow for correlated/entangled fractions of photons, as long as they stay microscopic, i.e. do not scale with N_i . The corresponding terms then factor out in front of the exponentials in (13,18) and the formation of the spectrum broadcast states is not affected.
- [24] If the difference $\overline{\tau_D}/\alpha - \overline{\tau_D}$ is sufficiently large, then

for $\overline{\tau_D} \ll t < \overline{\tau_D}/\alpha$ the state $\varrho_{S:FE}(t)$ is approximately a CQ state, whose mutual information is given by the Holevo quantity (A. S. Holevo, *Probl. Inform. Transm.* **9**, 177 (1973)): $I[\varrho_{S:FE}(t)] = S_{VN}[\sum_i p_i \varrho_i^{mac}(t)^{\otimes fM}] - (fM) \sum_i p_i S_{VN}[\varrho_i^{mac}(t)]$.

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Supplemental Material

The decoherence factor for mixed states

Here we calculate $|\text{Tr} \mathbf{S}_1 \varrho_0^{ph} \mathbf{S}_2^\dagger|$ in the leading order in the box size $1/L$ for the mixed states (3):

$$\begin{aligned} |\text{Tr} \mathbf{S}_1 \varrho_0^{ph} \mathbf{S}_2^\dagger|^2 &= \sum_{\vec{k}, \vec{k}'} p(\vec{k}) p(\vec{k}') \langle \vec{k} | \mathbf{S}_2^\dagger \mathbf{S}_1 \vec{k} \rangle \langle \vec{k}' | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle \\ &\cong \sum_{\vec{k}, \vec{k}'} p(\vec{k}) p(\vec{k}') \left(1 + \frac{iA_{\vec{k}}}{L^2} - \frac{B_{\vec{k}}}{L^2} \right) \left(1 - \frac{iA_{\vec{k}'}}{L^2} - \frac{B_{\vec{k}'}}{L^2} \right) \\ &\cong \sum_{\vec{k}, \vec{k}'} p(\vec{k}) p(\vec{k}') \left(1 + \frac{iA_{\vec{k}}}{L^2} - \frac{iA_{\vec{k}'}}{L^2} - \frac{B_{\vec{k}}}{L^2} - \frac{B_{\vec{k}'}}{L^2} \right) \quad (24) \\ &= 1 - 2 \sum_{\vec{k}} p(\vec{k}) \frac{B_{\vec{k}}}{L^2}, \quad (25) \end{aligned}$$

where we used Eq. (11) keeping only the terms up to $1/L^2$ and introduced $A_{\vec{k}} \equiv \frac{8}{3} \pi \Delta x k^5 \tilde{a}^6 \cos \Theta_{\vec{k}}$, $B_{\vec{k}} \equiv \frac{2\pi}{15} \Delta x^2 k^6 \tilde{a}^6 (3 + 11 \cos^2 \Theta_{\vec{k}})$. This gives in the leading order:

$$|\text{Tr} \mathbf{S}_1 \varrho_0^{ph} \mathbf{S}_2^\dagger| = \sqrt{1 - 2 \sum_{\vec{k}} p(\vec{k}) \frac{B_{\vec{k}}}{L^2}} \cong 1 - \sum_{\vec{k}} p(\vec{k}) \frac{B_{\vec{k}}}{L^2}, \quad (26)$$

leading to Eq. (12).

Calculation of $B(\varrho_1^{mic}, \varrho_2^{mic})$

We calculate the Bhattacharyya coefficient [1] $B(\varrho_1^{mic}, \varrho_2^{mic})$ (defined in Eq. (10)) for the individual photon states (micro-states) $\varrho_i^{mic} \equiv \mathbf{S}_i \varrho_0^{ph} \mathbf{S}_i^\dagger$ for general momentum-diagonal initial states (3) (our calculation is partially similar to that of Ref. [16]). Let:

$$\sqrt{\varrho_1^{mic}} \varrho_2^{mic} \sqrt{\varrho_1^{mic}} \equiv \mathbf{S}_1 \left(\sum_{\vec{k}, \vec{k}'} M_{\vec{k}\vec{k}'} |\vec{k}\rangle \langle \vec{k}'| \right) \mathbf{S}_1^\dagger, \quad (27)$$

where we have introduced a matrix:

$$M_{\vec{k}\vec{k}'} \equiv \sqrt{p(\vec{k}) p(\vec{k}')} \sum_{\vec{k}''} p(\vec{k}'') \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}'' \rangle \langle \vec{k}' | \mathbf{S}_2^\dagger \mathbf{S}_1 \vec{k}'' \rangle. \quad (28)$$

By Eq. (3) it is supported in the sector (2), and we diagonalize it in the leading order in $1/L$. For that, we first decompose matrix elements $M_{\vec{k}\vec{k}''}$ in $1/L$ and keep the leading terms only. Let us write:

$$\mathbf{S}_1^\dagger \mathbf{S}_2 = \mathbf{1} - (\mathbf{1} - \mathbf{S}_1^\dagger \mathbf{S}_2) \equiv \mathbf{1} - b. \quad (29)$$

Matrix elements of b between vectors satisfying (2) are of the order of $1/L$ at most: i) the diagonal elements $b_{\vec{k}\vec{k}} = 1 - \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k} \rangle = O(1/L^2)$ by Eq. (11); ii) the off-diagonal elements are determined by the unitarity of $\mathbf{S}_1^\dagger \mathbf{S}_2$ and the order of the diagonal ones: $1 = |\langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k} \rangle|^2 + \sum_{\vec{k}' \neq \vec{k}} |\langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle|^2 = 1 - O(1/L^2) + \sum_{\vec{k}' \neq \vec{k}} |b_{\vec{k}\vec{k}'}|^2$ for any fixed \vec{k} satisfying (2) (there is a single sum here), where we again used Eq. (11). Hence:

$$\forall \vec{k}: \sum_{\vec{k}' \neq \vec{k}} |b_{\vec{k}\vec{k}'}|^2 = \sum_{\vec{k}' \neq \vec{k}} |\langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle|^2 = O\left(\frac{1}{L^2}\right). \quad (30)$$

As a byproduct, from (30) it follows also that in the energy sector (2), $\mathbf{S}_1 \cong \mathbf{S}_2$ in the strong operator topology: $\|(\mathbf{S}_1 - \mathbf{S}_2)|\phi\rangle\|^2 = \|b|\phi\rangle\|^2 \xrightarrow{\text{therm.}} 0$ for any $|\phi\rangle$ from the subspace defined by (2). From Eqs. (11,28,29,30) we obtain in the leading order:

$$\begin{aligned} M_{\vec{k}\vec{k}''} &= p(\vec{k})^2 \delta_{\vec{k}\vec{k}''} - p(\vec{k})^{3/2} \sqrt{p(\vec{k}'')} b_{\vec{k}''\vec{k}}^* \\ &\quad - p(\vec{k}'')^{3/2} \sqrt{p(\vec{k})} b_{\vec{k}\vec{k}''} + O\left(\frac{1}{L^4}\right). \quad (31) \end{aligned}$$

The first term is non-negative and is of the order of unity, while the rest is of the order $1/L$ and forms a Hermitian matrix. We can thus calculate the desired eigenvalues $m(\vec{k})$ of $M_{\vec{k}\vec{k}''}$ using standard, stationary perturbation theory of quantum mechanics, treating the terms with the matrix b as a small perturbation. Assuming a generic non-degenerate situation (the measure $p(\vec{k})$ in Eq. (3) is injective), we obtain:

$$m(\vec{k}) = p(\vec{k})^2 \left(1 - b_{\vec{k}\vec{k}}^* - b_{\vec{k}\vec{k}} \right) + O\left(\frac{1}{L^4}\right), \quad (32)$$

and finally:

$$\begin{aligned} B(\varrho_1^{mic}, \varrho_2^{mic}) &= \text{Tr} \sqrt{\sqrt{\varrho_1^{mic}} \varrho_2^{mic} \sqrt{\varrho_1^{mic}}} = \text{Tr} \sqrt{M} \\ &\cong \sum_{\vec{k}} p(\vec{k}) \sqrt{1 - 2 \text{Re} b_{\vec{k}\vec{k}}} \cong \sum_{\vec{k}} p(\vec{k}) (1 - \text{Re} b_{\vec{k}\vec{k}}) \quad (33) \\ &= 1 + \frac{1}{2} \sum_{\vec{k}} \left(\frac{M_{\vec{k}\vec{k}}}{p(\vec{k})} - p(\vec{k}) \right) = \frac{1}{2} + \sum_{\vec{k}} \frac{p(\vec{k})}{2} |\langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k} \rangle|^2 \\ &\quad + \sum_{\vec{k}} \sum_{\vec{k}' \neq \vec{k}} \frac{p(\vec{k})}{2} |\langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle|^2 \equiv 1 - \frac{\bar{\eta} - \bar{\eta}'}{L^2}, \quad (34) \end{aligned}$$

where we have used Eqs. (27,32,11,28) in the respective order, and introduced:

$$\bar{\eta} \equiv \frac{L^2}{2} \left(1 - \sum_{\vec{k}} p(\vec{k}) \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k} \rangle \right|^2 \right) \cong \left(\frac{N}{\tau_D V} c \right)^{-1} \quad (35)$$

$$\bar{\eta}' \equiv \frac{L^2}{2} \sum_{\vec{k}} \sum_{\vec{k}' \neq \vec{k}} p(\vec{k}) \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle \right|^2 \quad (36)$$

(in Eq. (35) we have used Eqs. (12,13)). We note that $\bar{\eta}, \bar{\eta}'$ are of the order of unity in $1/L$ by Eqs. (13,30).

Isotropic illumination

We show that for a completely isotropic illumination the macro-states never orthogonalize and hence such an environment carries no information on the sphere's localization, as it intuitively should not (cf. Refs. [2, 4]). To prove it, we calculate

$$\sum_{\vec{k}, \vec{k}'} p(\vec{k}) \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle \right|^2 \quad (37)$$

for $p(\vec{k}) \equiv p(k)(1/4\pi)$. Or more precisely, since we are working in the box normalization, the measure is

$$p(\vec{k}) \equiv p(k)(1/\Omega_k), \quad (38)$$

where Ω_k is the number of the discrete box states $|\vec{k}\rangle$ with the fixed length $k = |\vec{k}|$. In the continuous limit Ω_k approaches $4\pi k^2$. As the scattering is by assumption elastic, matrix elements $\langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle$ are non-zero only for the equal lengths $k = k'$ and hence:

$$\mathbf{S}_1^\dagger \mathbf{S}_2 \equiv \bigoplus_k U_k, \quad U_k^\dagger U_k = \mathbf{1}_k. \quad (39)$$

Decomposing the summations over \vec{k}, \vec{k}' into the sums over the lengths k, k' and the directions $\vec{n}(k), \vec{n}(k')$ and using (39), we obtain:

$$\sum_{\vec{k}, \vec{k}'} p(\vec{k}) \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle \right|^2 = \sum_k \frac{p(k)}{\Omega_k} \sum_{\vec{n}(k), \vec{n}(k')} \left| \langle \vec{k} | \mathbf{S}_1^\dagger \mathbf{S}_2 \vec{k}' \rangle \right|^2 = \quad (40)$$

$$\sum_k \frac{p(k)}{\Omega_k} \text{Tr} \left(P_k \mathbf{S}_1^\dagger \mathbf{S}_2 P_k \mathbf{S}_2 \mathbf{S}_1^\dagger \right) = \quad (41)$$

$$\sum_k p(k) \frac{\text{Tr} P_k}{\Omega_k} = 1, \quad (42)$$

where $P_k \equiv \sum_{\vec{n}(k)} |\vec{k}\rangle \langle \vec{k}|$ is a projector onto the subspace of a fixed length k , and hence $\text{Tr} P_k = \Omega_k$. Comparing with Eq. (34), Eq. (42) leads to $\bar{\eta} - \bar{\eta}' = 0$, and since by definition $\alpha = (\bar{\eta} - \bar{\eta}')/\bar{\eta}$, to $\alpha = 0$.

Derivation of the quantum Darwinism relation (21)

We show that Eq. (21) follows from the mechanisms of i) decoherence, Eq. (9), and ii) distinguishability, Eq. (10) and is thus a consequence of the state information broadcasting process.

We generalize to mixed environment states the similar calculations from Ref. [5]. Let the post-interaction $S : fE$ state for a fixed, finite box L and time t be $\varrho_{S:fE}(L, t)$. It is given by Eqs. (7,8) and now we explicitly indicate the dependence on L in the notation. Then:

$$|H_S - I[\varrho_{S:fE}(L, t)]| \leq \left| I[\varrho_{S:fE}(L, t)] - I[\varrho_{S:fE}^{i=j}(L, t)] \right| \quad (43)$$

$$+ \left| H_S - I[\varrho_{S:fE}^{i=j}(L, t)] \right|, \quad (44)$$

where $\varrho_{S:fE}^{i=j}(L, t)$ is the decohered part of $\varrho_{S:fE}(L, t)$, given by Eq. (7). We first bound the difference (43), decomposing the mutual information using conditional information $S_{\text{vN}}(\varrho_{S:fE}|\varrho_{fE}) \equiv S_{\text{vN}}(\varrho_{S:fE}) - S_{\text{vN}}(\varrho_{fE})$:

$$I(\varrho_{S:fE}) = S_{\text{vN}}(\varrho_S) - S_{\text{vN}}(\varrho_{S:fE}|\varrho_{fE}), \quad (45)$$

so that:

$$\begin{aligned} & \left| I[\varrho_{S:fE}(L, t)] - I[\varrho_{S:fE}^{i=j}(L, t)] \right| \leq \\ & \left| S_{\text{vN}}[\varrho_S(L, t)] - S_{\text{vN}}[\varrho_S^{i=j}(L, t)] \right| + \\ & \left| S_{\text{vN}}[\varrho_{S:fE}(L, t)|\varrho_{fE}(L, t)] - S_{\text{vN}}[\varrho_{S:fE}^{i=j}(L, t)|\varrho_{fE}^{i=j}(L, t)] \right|. \end{aligned} \quad (46)$$

From Eq. (2), the total $S : fE$ Hilbert space is finite-dimensional for a finite L, t : there are $fN_t = fL^2(N/V)ct$ photons (cf. Eq. (4)) and the number of modes of each photon is approximately $(4\pi/3)(L/2\pi\Delta x)^3$. Hence, the total dimension is $2 \times L^2 f(N/V)ct \times (1/6\pi^2)(L/\Delta x)^3 < \infty$ and we can use the Fannes-Audenaert [7] and the Alicki-Fannes [8] inequalities to bound (46) and (47) respectively. For (46) we obtain:

$$\begin{aligned} & \left| S_{\text{vN}}[\varrho_S(L, t)] - S_{\text{vN}}[\varrho_S^{i=j}(L, t)] \right| \\ & \leq \frac{1}{2} \epsilon_E(L, t) \log(d_S - 1) + h \left[\frac{\epsilon_E(L, t)}{2} \right], \end{aligned} \quad (48)$$

where $h(\epsilon) \equiv -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon)$ is the binary Shannon entropy and:

$$\epsilon_E(L, t) \equiv \|\varrho_S(L, t) - \varrho_S^{i=j}(L, t)\|_{\text{tr}} \quad (49)$$

$$= \|\varrho_S^{i \neq j}(L, t)\|_{\text{tr}} \cong 2|c_{12}| \left[1 - \frac{1}{c\tau_D L^2} \left(\frac{N}{V} \right)^{-1} \right]^{L^2 \frac{N}{V} ct} \quad (50)$$

with $c_{12} \equiv \langle \vec{x}_1 | \varrho_0^S | \vec{x}_2 \rangle$, where we have used the same reasoning (13), but with $f = 0$. For (47) the same reasoning and the Alicki-Fannes inequality give:

$$\left| S_{\text{vN}} [\varrho_{S:fE}(L, t) | \varrho_{fE}(L, t)] - S_{\text{vN}} [\varrho_{S:fE}^{i=j}(L, t) | \varrho_{fE}^{i=j}(L, t)] \right| \leq 4\epsilon_{fE}(L, t) \log d_S + 2h[\epsilon_{fE}(L, t)], \quad (51)$$

with:

$$\epsilon_{fE}(L, t) \equiv \|\varrho_{S:fE}(L, t) - \varrho_{S:fE}^{i=j}(L, t)\|_{\text{tr}} \quad (52)$$

$$= \|\varrho_{S:fE}^{i \neq j}(L, t)\|_{\text{tr}} \quad (53)$$

$$\cong 2|c_{12}| \left[1 - \frac{1}{c_{\overline{TD}} L^2} \left(\frac{N}{V} \right)^{-1} \right]^{L^2(1-f)\frac{N}{V}ct} \quad (54)$$

Above L, t are big enough so that $\epsilon_E(L, t), \epsilon_{fE}(L, t) < 1$. Eqs. (46-54) give an upper bound on the difference (43) in terms of the decoherence speed (9).

To bound the "orthogonalization" part (44) (see Ref. [4] for a related analysis), we note that since $\varrho_{S:fE}^{i=j}(L, t)$ is a CQ-state (cf. Eq. (7)), its mutual information is given by the Holevo quantity [6]:

$$I[\varrho_{S:fE}^{i=j}(L, t)] = \chi\{p_i, \varrho_i^{\text{mac}}(t)^{\otimes fM}\}. \quad (55)$$

From the Holevo Theorem it is bounded by [24]:

$$I_{\text{max}}(t) \leq \chi\{p_i, \varrho_i^{\text{mac}}(t)^{\otimes fM}\} \leq H(\{p_i\}) \equiv H_S, \quad (56)$$

where $I_{\text{max}}(t) \equiv \max_{\mathcal{E}} I[p_i \pi_{j|i}^{\mathcal{E}}(t)]$ is the fixed time maximal mutual information, extractable through generalized measurements $\{\mathcal{E}_j\}$ on the ensemble $\{p_i, \varrho_i^{\text{mac}}(t)^{\otimes fM}\}$, and the conditional probabilities read:

$$\pi_{j|i}^{\mathcal{E}}(t) \equiv \text{Tr}[\mathcal{E}_j \varrho_i^{\text{mac}}(t)^{\otimes fM}] \quad (57)$$

(here and below i labels the states, while j the measurement outcomes). We now relate $I_{\text{max}}(t)$ to the generalized overlap $B[\varrho_1^{\text{mac}}(t)^{\otimes fM}, \varrho_2^{\text{mac}}(t)^{\otimes fM}]$ (cf. Eq. (10)), which we have calculated in Eq. (18). Using the method of Ref. [1], slightly modified to unequal a priori probabilities p_i , we obtain for an arbitrary measurement \mathcal{E} :

$$I(\pi_{j|i}^{\mathcal{E}} p_i) = I(\pi_{i|j}^{\mathcal{E}} \pi_j^{\mathcal{E}}) = H(\{p_i\}) - \sum_{j=1,2} \pi_j^{\mathcal{E}} h(\pi_{1|j}^{\mathcal{E}}) \quad (58)$$

$$\geq H(\{p_i\}) - 2 \sum_{j=1,2} \pi_j^{\mathcal{E}} \sqrt{\pi_{1|j}^{\mathcal{E}} (1 - \pi_{1|j}^{\mathcal{E}})} \quad (59)$$

$$= H(\{p_i\}) - 2\sqrt{p_1 p_2} \sum_{j=1,2} \sqrt{\pi_{j|1}^{\mathcal{E}} \pi_{j|2}^{\mathcal{E}}}, \quad (60)$$

where we have first used Bayes Theorem $\pi_{i|j}^{\mathcal{E}} = (p_i/\pi_j^{\mathcal{E}}) \pi_{j|i}^{\mathcal{E}}$, $\pi_j^{\mathcal{E}} \equiv \sum_i \pi_{j|i}^{\mathcal{E}} p_i = \text{Tr}(\mathcal{E}_j \sum_i \varrho_i)$, then the fact that we have only two states: $\pi_{2|j}^{\mathcal{E}} = 1 - \pi_{1|j}^{\mathcal{E}}$, so that

$H(\pi_{j|i}^{\mathcal{E}}) = h(\pi_{1|j}^{\mathcal{E}})$, and finally $h(p) \leq 2\sqrt{p(1-p)}$. On the other hand, $B(\varrho_1, \varrho_2) = \min_{\mathcal{E}} \sum_j \sqrt{\pi_{j|1}^{\mathcal{E}} \pi_{j|2}^{\mathcal{E}}} [1]$. Denoting the optimal measurement by $\mathcal{E}_*^B(t)$ and recognizing that $H(\{p_i\}) = H_S$, we obtain:

$$I_{\text{max}}(t) \geq I[p_i \pi_{j|i}^{\mathcal{E}_*^B(t)}(t)] \geq H_S - \quad (61)$$

$$-2\sqrt{p_1 p_2} B[\varrho_1^{\text{mac}}(t)^{\otimes fM}, \varrho_2^{\text{mac}}(t)^{\otimes fM}] \quad (62)$$

$$= H_S - 2\sqrt{p_1 p_2} B[\varrho_1^{\text{mac}}(t), \varrho_2^{\text{mac}}(t)]^{fM} \quad (63)$$

Inserting the above into the bounds (56) gives the desired upper bound on the difference (44):

$$\left| H_S - I[\varrho_{S:fE}^{i=j}(L, t)] \right| \leq 2\sqrt{p_1 p_2} B[\varrho_1^{\text{mac}}(t), \varrho_2^{\text{mac}}(t)]^{fM} \quad (64)$$

where the generalized overlap is given by Eq. (18):

$$B[\varrho_1^{\text{mac}}(t), \varrho_2^{\text{mac}}(t)] \cong \left[1 - \frac{\alpha}{c_{\overline{TD}} L^2} \left(\frac{N}{V} \right)^{-1} \right]^{L^2 m \frac{N}{V} ct} \quad (65)$$

Gathering all the above facts together finally leads to a bound on $|H_S - I[\varrho_{S:fE}(L, t)]|$ in terms of the speed of i) decoherence (9) and ii) distinguishability (10):

$$|H_S - I[\varrho_{S:fE}(L, t)]| \leq h\left[\frac{\epsilon_E(L, t)}{2}\right] + 2h[\epsilon_{fE}(L, t)] + 4\epsilon_{fE}(L, t) \log 2 + 2\sqrt{p_1 p_2} B[\varrho_1^{\text{mac}}(t), \varrho_2^{\text{mac}}(t)]^{fM}, \quad (66)$$

where $\epsilon_E(L, t)$, $\epsilon_{fE}(L, t)$, $B[\varrho_1^{\text{mac}}(t), \varrho_2^{\text{mac}}(t)]$ are given by Eqs. (50), (54), and (65) respectively. Choosing L, t big enough so that $\epsilon_E(L, t), \epsilon_{fE}(L, t) \leq 1/2$ (when the binary entropy $h(\cdot)$ is monotonically increasing), we remove the unphysical box and obtain an estimate on the speed of convergence of $I[\varrho_{S:fE}(L, t)]$ to H_S :

$$\lim_{L \rightarrow \infty} |H_S - I[\varrho_{S:fE}(L, t)]| \leq h\left(|c_{12}| e^{-\frac{t}{\overline{TD}}}\right) \quad (68)$$

$$+ 2h\left(2|c_{12}| e^{-\frac{(1-f)}{\overline{D}} t}\right) + 8|c_{12}| e^{-\frac{(1-f)}{\overline{D}} t} \log 2 \quad (69)$$

$$+ 2\sqrt{p_1 p_2} e^{-\frac{\alpha f}{\overline{D}} t}. \quad (70)$$

This finishes the derivation of the Quantum Darwinism condition (21).

We note that the result (66,67) is in fact a general statement, valid in any model where: i) the system S is effectively a qubit; ii) the system-environment interaction is of a environment-symmetric, controlled-unitary type:

Theorem 1 *Let a two-dimensional quantum system S interact with N identical environments, each described by a finite-dimensional Hilbert space, through a controlled-unitary interaction:*

$$U(t) \equiv \sum_{i=1,2} |i\rangle\langle i| \otimes U_i(t)^{\otimes N}. \quad (71)$$

Let the initial state be $\varrho_{S:E}(0) = \varrho_0^S \otimes (\varrho_0^E)^{\otimes N}$ and $\varrho_{S:E}(t) \equiv U(t)\varrho_{S:E}(0)U(t)^\dagger$. Then for any $0 < f < 1$ and t big enough:

$$|H(\{p_i\}) - I[\varrho_{S:fE}(t)]| \leq h \left[\frac{\epsilon_E(t)}{2} \right] + 2h[\epsilon_{fE}(t)] + \quad (72)$$

$$4\epsilon_{fE}(t) \log 2 + 2\sqrt{p_1 p_2} B[\varrho_1^{mac}(t), \varrho_2^{mac}(t)]^{fN}, \quad (73)$$

where:

$$p_i \equiv \langle i | \varrho_0^S | i \rangle, \varrho_i(t) \equiv U_i(t) \varrho_0^E U_i(t)^\dagger, \quad (74)$$

$$\epsilon_E(t) \equiv \|\varrho_S(t) - \varrho_S^{i=j}\|_{tr}, \quad (75)$$

$$\epsilon_{fE}(t) \equiv \|\varrho_{S:fE}(t) - \varrho_{S:fE}^{i=j}(t)\|_{tr}. \quad (76)$$

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